

Appendix 1

Detailed derivation of result

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2 In this appendix we provide a more detailed derivation of the result of Eq. 6 in
3 the main text, and also discuss the accuracy of the approximation used in the
4 last step of the derivation. To express the demographic variance by the
5 autocorrelations in population growth, we consider the variance of population
6 size over time. First, we consider the case of a population where population
7 structure is ignored.

8 **Stochastic growth assuming no population structure**

9 In the model where the population structure is ignored, all variation among
10 individuals in demographic processes like survival and reproduction is assumed to
11 be entirely random. Note however that the underlying dynamical process we

12 consider is still that of a structured population (only this model applied to it
 13 does not account for the structure). The contribution of an individual to next
 14 year's population growth rate is a random variable with mean λ and variance σ_d^{2*}
 15 (the demographic variance assuming no structure). Next year's population size is
 16 given by $N_{t+1} = \Lambda_t N_t$, where Λ_t is a stochastic annual growth rate (not to be
 17 confused with the long-term population growth rate of a stochastic model, which
 18 is often denoted 'stochastic growth rate'). The annual growth rate can be
 19 decomposed as $\Lambda_t = \lambda + \epsilon_t/N_t$, where $\lambda = E[\Lambda_t]$, while the stochastic element ϵ_t
 20 has expectation 0 and variance $N_t \sigma_d^{2*}$ (conditional on the current population size
 21 N_t). On log scale, the conditional variance over one time step is given by

$$\begin{aligned}
 \text{Var}(\ln N_{t+1}|N_t) &= \text{Var}\left(\ln\left(\lambda + \frac{\epsilon_t}{N_t}\right) + \ln N_t \middle| N_t\right) \\
 &= \text{Var}\left(\ln\left[\lambda N_t \left(1 + \frac{\epsilon_t}{\lambda N_t}\right)\right] \middle| N_t\right) \\
 &= \text{Var}\left(\ln \lambda + \ln N_t + \ln\left(1 + \frac{\epsilon_t}{\lambda N_t}\right) \middle| N_t\right) \\
 &= \text{Var}\left(\ln\left(1 + \frac{\epsilon_t}{\lambda N_t}\right) \middle| N_t\right) \\
 &\approx \text{Var}\left(\frac{\epsilon_t}{\lambda N_t} \middle| N_t\right) \\
 &= \frac{\sigma_d^{2*}}{\lambda^2 N_t},
 \end{aligned}$$

22 where the approximation is a first-order Taylor approximation of $\ln\left(1 + \frac{\epsilon_t}{\lambda N_t}\right)$
 23 around $\frac{\epsilon_t}{\lambda N_t} = 0$. This approximation holds if the term $\frac{\epsilon_t}{\lambda N_t}$ is small. Using a
 24 second-order Taylor approximation of the same expression, the conditional
 25 expectation is given by

$$\begin{aligned}
 \mathbb{E}[\ln N_{t+1}|N_t] &= \mathbb{E}[\ln(\lambda + \epsilon_t/N_t) + \ln N_t|N_t] \\
 &= \mathbb{E}\left[\ln \lambda + \ln N_t + \ln\left(1 + \frac{\epsilon_t}{\lambda N_t}\right) | N_t\right] \\
 &\approx \ln \lambda + \ln N_t + \mathbb{E}\left[\frac{\epsilon_t}{\lambda N_t}\right] - \mathbb{E}\left[\frac{\epsilon_t^2}{2\lambda^2 N_t^2}\right] \\
 &= \ln \lambda + \ln N_t - \frac{\sigma_d^{2*}}{2\lambda^2 N_t}.
 \end{aligned}$$

26 The unconditional variance in population growth over one time step
 27 ($\Delta \ln N_t = \ln N_{t+1} - \ln N_t$) is obtained by applying the law of total variance and
 28 using the above expressions, and is given by

$$\begin{aligned}
\text{Var}(\Delta \ln N_t) &= \text{E}[\text{Var}(\Delta \ln N_t | N_t)] + \text{Var}(\text{E}[\Delta \ln N_t | N_t]) \\
&= \text{E}[\text{Var}(\ln N_{t+1} | N_t)] + \text{Var}(\text{E}[\ln N_{t+1} - \ln N_t | N_t]) \\
&= \text{E} \left[\frac{\sigma_d^{2*}}{\lambda^2 N_t} \right] + \text{Var} \left(\ln \lambda + \ln N_t - \frac{\sigma_d^{2*}}{2\lambda^2 N_t} \right) \\
&\approx \frac{\sigma_d^{2*}}{\lambda^2} \text{E} [N_t^{-1}] + \text{Var} \left(\ln \lambda - \frac{\sigma_d^{2*}}{2\lambda^2 N_t} \right) \\
&= \frac{\sigma_d^{2*}}{\lambda^2} \left(\text{E} [N_t^{-1}] + \frac{\sigma_d^{2*}}{4\lambda^2} \text{Var} (N_t^{-1}) \right) \\
&= \frac{\sigma_d^{2*}}{\lambda^2} \left(\text{E} [N_t^{-1}] + \frac{\sigma_d^{2*}}{4\lambda^2} \left(\text{E} [N_t^{-2}] - \text{E} [N_t^{-1}]^2 \right) \right) \\
&\approx \frac{\sigma_d^{2*}}{\lambda^2} \text{E} [N_t^{-1}].
\end{aligned}$$

29 Using the formula for the variance of a sum of random variables (and that
30 $\text{Var}(\ln N_0 | N_0) = 0$), the variance of $\ln N_t$ after t time steps is given by

$$\begin{aligned}
\text{Var}(\ln N_t | N_0) &= \text{Var} (\Delta \ln N_{t-1} + \Delta \ln N_{t-2} + \cdots + \Delta \ln N_0 + \ln N_0) \\
&= \text{Var} \left(\sum_{j=0}^{t-1} \Delta \ln N_j \right) \\
&= \sum_{i,j=0}^{t-1} \text{Cov} (\Delta \ln N_i, \Delta \ln N_j) \\
&= \sum_{i,j=0}^{t-1} \text{Corr} (\Delta \ln N_i, \Delta \ln N_j) \sqrt{\text{Var}(\Delta \ln N_i)} \sqrt{\text{Var}(\Delta \ln N_j)}.
\end{aligned}$$

31 Inserting the above expressions for the variance of $\Delta \ln N_t$ and rearranging the
 32 sum, we obtain the expression for the variance of population size

$$\begin{aligned}
 \text{Var}(\ln N_t | N_0) &\approx \sum_{i,j=0}^{t-1} \text{Corr}(\Delta \ln N_i, \Delta \ln N_j) \frac{\sigma_d^{2*}}{\lambda^2} \sqrt{\text{E}[N_i^{-1}] \text{E}[N_j^{-1}]} \\
 &= \frac{\sigma_d^{2*}}{\lambda^2} \left(\sum_{j=0}^{t-1} \text{E}[N_j^{-1}] + \sum_{i \neq j} \text{Corr}(\Delta \ln N_i, \Delta \ln N_j) \sqrt{\text{E}[N_i^{-1}] \text{E}[N_j^{-1}]} \right) \\
 &= \frac{\sigma_d^{2*}}{\lambda^2} \left(\sum_{j=0}^{t-1} \text{E}[N_j^{-1}] + 2 \sum_{j=1}^{t-1} \sum_{i=0}^{t-j-1} \rho_j \sqrt{\text{E}[N_i^{-1}] \text{E}[N_{i+j}^{-1}]} \right) \\
 &= \frac{\sigma_d^{2*}}{\lambda^2} \left(\sum_{j=0}^{t-1} \text{E}[N_j^{-1}] + 2 \sum_{j=1}^{t-1} \rho_j \sum_{i=0}^{t-j-1} \sqrt{\text{E}[N_i^{-1}] \text{E}[N_{i+j}^{-1}]} \right). \quad (\text{A1})
 \end{aligned}$$

33 Variance in population growth of the structured population 34 model

35 We now apply a model accounting for the population structure and first consider
 36 the process of the total reproductive value V_t , which has approximately no
 37 autocorrelation (Engen et al. 2007, 2009). Using a similar method as for the
 38 unstructured model above, but this time for the total reproductive value, the
 39 variance after t time steps is given by

$$\text{Var}(\ln V_t | V_0) = \text{Var} \left(\sum_{j=0}^{t-1} \Delta \ln V_j \right) \approx \frac{\sigma_d^2}{\lambda^2} \sum_{j=0}^{t-1} \text{E} [V_j^{-1}] \approx \frac{\sigma_d^2}{\lambda^2} \sum_{j=0}^{t-1} \text{E} [N_j^{-1}], \quad (\text{A2})$$

40 where the last approximation uses $N_j \approx V_j$. Putting this together with equation
 41 (A1), we obtain

$$\begin{aligned} \frac{\sigma_d^2}{\lambda^2} \sum_{j=0}^{t-1} \text{E} [N_j^{-1}] &\approx \frac{\sigma_d^{2*}}{\lambda^2} \sum_{j=0}^{t-1} \text{E} [N_j^{-1}] + 2 \sum_{j=1}^{t-1} \rho_j \sum_{i=0}^{t-j-1} \sqrt{\text{E} [N_i^{-1}] \text{E} [N_{i+1}^{-1}]} \\ \sigma_d^2 &\approx \sigma_d^{2*} \left(1 + 2 \sum_{j=1}^{t-1} \rho_j \left[\sum_{i=0}^{t-j-1} \sqrt{\text{E} [N_i^{-1}] \text{E} [N_{i+j}^{-1}]} / \left[\sum_{k=0}^{t-1} \text{E} [N_k^{-1}] \right] \right] \right) \\ &\approx \sigma_d^{2*} \left(1 + 2 \sum_{j=1}^{t-1} \rho_j \right), \end{aligned} \quad (\text{A3})$$

42 where the last approximation is discussed in the next section. Finally, taking the
 43 limit as $t \rightarrow \infty$, we get the expression

$$\sigma_d^2 \approx \sigma_d^{2*} \left(1 + 2 \sum_{j=1}^{\infty} \rho_j \right), \quad (\text{A4})$$

44 which corresponds to Eq. 6 in the main text.

45 Evaluating the accuracy of the approximation

The last approximation in Eq. A3 is based on the assumption that for each time lag j , the autocorrelation ρ_j is close to zero and/or the fraction

$$\alpha_j = \left[\sum_{i=0}^{T-j-1} \sqrt{\mathbb{E}[N_i^{-1}] \mathbb{E}[N_{i+j}^{-1}]} \right] / \left[\sum_{k=0}^{T-1} \mathbb{E}[N_k^{-1}] \right] \quad (\text{A5})$$

48 is close to 1 (where T is large enough to cover the relevant range of
 49 autocorrelation). For $j = 0$ the expression is $\alpha_j = 1$, and for increasing lags this
 50 fraction will decline towards zero (see example below). At the same time, the
 51 autocorrelation function ρ_j approaches zero, so that the approximation $\rho_j \alpha_j \approx \rho_j$ in
 Eq. A3 will be fairly accurate in most cases.

53 We evaluated the accuracy of this approximation using simulations, for the
 54 two hypothetical examples presented in the main text. These examples consider a
 55 population living in two distinct habitats, where offspring disperse randomly
 56 between them (with probability 0.5 of dispersing to the other habitat), and then
 57 remain in a habitat for life. In one example we assumed different fecundity in the
 58 two habitats (Poisson distributed with mean $b_1 = 0.7$ and $b_2 = 0.2$), while the
 59 survival probability was the same ($s_1 = s_2 = 0.6$). For the second example we
 60 assumed that survival probabilities were different ($s_1 = 0.7$ and $s_2 = 0.2$) while
 61 the fecundity was the same (Poisson distributed with mean $b_1 = b_2 = 0.5$). For

62 each example we generated 50 000 realizations of the population growth process,
63 where each simulation was for 300 time steps and starting from 100 individuals
64 at the stable distribution. From the simulated time series of $\Delta \ln N_t$ we
65 calculated the sample autocorrelation function (same results as in the main text). 66
From the corresponding time series of $1/N_t$ we estimated the expression α_j in Eq. A5 across
time lag j .

68 The results are shown in Fig. A1, demonstrating that the approximation
69 is accurate at least in these examples. We also show how the estimated fraction
70 α_j declines with increasing time lag (the decline is slightly faster for the example
71 with different survival). Since the autocorrelation function ρ_j also approaches
72 zero, the resulting error from excluding α_j (i.e. assuming $\rho_j \alpha_j \approx \rho_j$) is not large
73 (compare the red and black lines in the lower panels).

74 We emphasize that these examples do not confirm that the approximation
75 will be accurate for all other cases. In general, we expect that it will be more
76 accurate for models where the autocorrelation function approaches zero relatively
77 fast, and for populations with a mean growth rate close to 1.

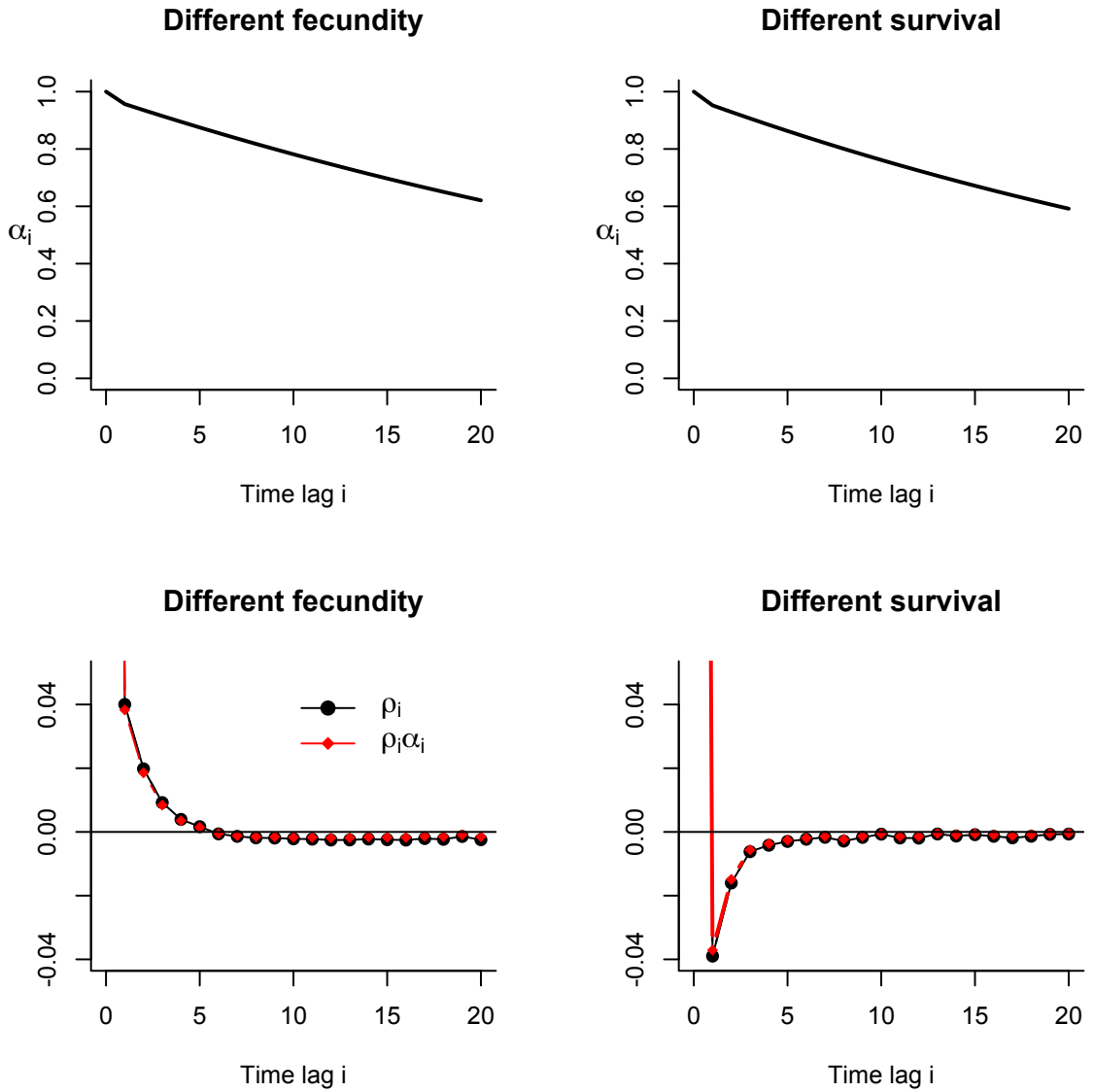


Figure A1. Upper panels show the estimated fraction α_i (Eq. A5), while lower panels show the the sample autocorrelation function ρ_i (black lines) together with the combined expression $\alpha_i \rho_i$ (red lines). Examples are from a hypothetical population assuming two distinct habitats affecting either the fecundity (left panels, fecundity is Poisson distributed with $b_1 = 0.7$ and $b_2 = 0.2$, survival probability is $s_1 = s_2 = 0.6$) or the survival (right panels, survival probabilities are $s_1 = 0.7$ and $s_2 = 0.2$, fecundity is $b_1 = b_2 = 0.5$). The estimates are based on 50 000 simulations over 300 time steps, starting from 100 individuals at the stable distribution.

78 **Appendix 2**

Autocorrelation function related to **underlying vital rates**

79

80 In this appendix we describe how to calculate the autocovariance function (and
81 thereby the autocorrelation function) of $\Delta \ln N_t$ from the underlying vital rates.

82 This is only possible if all relevant covariances between elements of the stochastic
83 matrix \mathbf{M}_t are known.

84 **Model description**

85 The population model is also described in the main text; here we provide some
86 additional definitions to elements of the stochastic projection matrix \mathbf{M}_t . We
87 consider a structured population of k stages, living in a constant environment (no
88 environmental stochasticity), where the population growth over one time step is
89 given by $\mathbf{n}_{t+1} = \mathbf{M}_t \mathbf{n}_t$. For simpler notation we will drop the time index t unless
90 necessary.

91 At a given time step, let S_j denote the number of individuals in stage j
92 that survive until next time step. Similarly, the total number of offspring
93 produced (before survival) that survive to the next time time step is denoted B_j .
94 The average survival in stage j is then a random variable given by $\bar{S}_j = \frac{S_j}{n_j}$, while
95 the average number of offspring produced per individual is given by $\bar{B}_j = \frac{B_j}{n_j}$.

96 Each time step the individuals and their offspring can also make a

97 transition to other stages (or remain in their current stage), and these transitions
 98 also contribute to the demographic stochasticity in population growth. Let R_{ij}
 99 denote the number of surviving individuals from class j that make a transition to
 100 class i . The probability that an individual from stage j makes this transition is
 101 r_{ij} , so that $E[R_{ij}|S_j] = S_j r_{ij}$, $\text{Var}[R_{ij}|S_j] = S_j r_{ij}(1 - r_{ij})$, and
 102 $\text{Cov}(R_{ij}, R_{lj}|S_j) = -S_j r_{ij} r_{lj}$ for $i \neq l$. For the offspring transitions we have
 103 similar expressions, letting Q_{ij} denote the number of offspring from stage j that
 104 make a transition to stage i . This occurs with probability q_{ij} for each offspring,
 105 so that $E[Q_{ij}|B_j] = B_j q_{ij}$, $\text{Var}[Q_{ij}|B_j] = B_j q_{ij}(1 - q_{ij})$, and
 106 $\text{Cov}(Q_{ij}, Q_{lj}|B_j) = -B_j q_{ij} q_{lj}$ for $i \neq l$.

107 Putting these factors together, an element M_{ij} of the matrix \mathbf{M} is given by

$$\begin{aligned}
 M_{ij} &= \frac{1}{n_j} S_j \frac{1}{S_j} R_{ij} + \frac{1}{n_j} B_j \frac{1}{B_j} Q_{ij} \\
 &= \frac{1}{n_j} (R_{ij} + Q_{ij}).
 \end{aligned}$$

108 Covariance of elements in \mathbf{M}

109 The covariances of different elements in \mathbf{M} can be used to find the autocovariance
 110 function of the process X_t derived in the next section. Because we assume that
 111 all individual contributions to population growth are independent, there is no

112 covariance between elements of different columns in \mathbf{M} . Also, there is no
 113 covariance between the transitions of adults (non-offspring) and offspring. To find
 114 the remaining non-zero covariances we use the law of conditional covariance and
 115 condition on the total number of offspring B_j and the total number of survivors
 116 S_j in class j . The covariance of elements from column j is then given by

$$\text{Cov}(M_{ij}, M_{lj}) = \text{E}[\text{Cov}(M_{ij}, M_{lj} | S_j, B_j)] + \text{Cov}(\text{E}[M_{ij} | S_j, B_j], \text{E}[M_{lj} | S_j, B_j]).$$

(B.1)

117 The first term of Eq. B.1 is given by

$$\begin{aligned} \text{E}[\text{Cov}(M_{ij}, M_{lj} | S_j, B_j)] &= \text{E} \left[\text{Cov} \left(\frac{1}{n_j} (R_{ij} + Q_{ij}), \frac{1}{n_j} (R_{lj} + Q_{lj}) \middle| S_j, B_j \right) \right] \\ &= \text{E} \left[\frac{1}{n_j^2} \text{Cov} (R_{ij}, R_{lj} | S_j) \right] + \text{E} \left[\frac{1}{n_j^2} \text{Cov} (Q_{ij}, Q_{lj} | B_j) \right] \\ &= \frac{1}{n_j} (s_j \alpha_{il,j} + b_j \beta_{il,j}), \end{aligned}$$

118 where

$$\alpha_{il,j} = \begin{cases} r_{ij}(1 - r_{ij}), & i = l \\ -r_{ij}r_{lj}, & i \neq l, \end{cases},$$

and

$$\beta_{il,j} = \begin{cases} q_{ij}(1 - q_{ij}), & i = l \\ -q_{ij}q_{lj}, & i \neq l. \end{cases}$$

119 The second term of Eq. B.1 is given by

$$\begin{aligned} \text{Cov}(\mathbb{E}[M_{ij}|S_j, B_j], \mathbb{E}[M_{lj}|S_j, B_j]) &= \frac{1}{n_j^2} \text{Cov}(S_j r_{ij} + B_j q_{ij}, S_j r_{lj} + B_j q_{lj}) \\ &= \frac{1}{n_j^2} [\text{Cov}(S_j r_{ij}, S_j r_{lj}) + \text{Cov}(B_j q_{ij}, B_j q_{lj}) + \text{Cov}(S_j r_{ij}, B_j q_{lj}) + \text{Cov}(S_j r_{lj}, B_j q_{ij})] \\ &= \frac{1}{n_j} [r_{ij}r_{lj}\sigma_{S_j}^2 + q_{il}q_{lj}\sigma_{B_j}^2 + r_{ij}q_{lj}\sigma_{BS_j}^2 + r_{lj}q_{ij}\sigma_{BS_j}^2]. \end{aligned}$$

120 Adding up the two terms and applying the approximation $n_j \approx u_j N$, we obtain

$$\text{Cov}(M_{ij}, M_{lj}) \approx \frac{1}{u_j N} [s_j \alpha_{il,j} + b_l \beta_{il,j} + r_{ij}r_{lj}\sigma_{S_j}^2 + q_{ij}q_{lj}\sigma_{B_j}^2 + r_{ij}q_{lj}\sigma_{BS_j}^2 + r_{lj}q_{ij}\sigma_{BS_j}^2]. \quad (\text{B.2})$$

121 **Autocovariance function of $\Delta \ln N$**

122 In this section we will derive the autocovariance function of the growth
 123 increments $\Delta \ln N$. To do this, we follow the method of Engen et al. (2007) and
 124 first consider the dynamics of the deviations between the population size and
 125 total reproductive value, defined as $\mathbf{x}_t = \mathbf{n}_t/V_t - \mathbf{u} = V_t^{-1}(\mathbf{I} - \mathbf{u}\mathbf{v})\mathbf{n}_t$, where \mathbf{I} is
 126 the identity matrix. The elements of the vector \mathbf{x}_t will fluctuate around zero as \mathbf{n}_t
 127 fluctuates around $\mathbf{u}V_t$. Letting $X_t = \mathbf{e}\mathbf{x}_t = N_t/V_t - 1$, where \mathbf{e} is a row vector of
 128 length k with elements equal to 1, we have that $N_t = (X_t + 1)V_t$ and thereby
 129 $\ln N_t \approx \ln V_t + X_t$, assuming X_t is small so that a first-order Taylor
 130 approximation of $\ln(1 + X_t)$ can be applied (Engen et al. 2007).

131 To study the dynamics of \mathbf{x}_t we first consider the dynamics of \mathbf{n}_t , given by
 132 $\mathbf{n}_{t+1} = \mathbf{K}\mathbf{n}_t + \varepsilon_t V_t(\mathbf{u} + \mathbf{x}_t) \approx \mathbf{K}\mathbf{n}_t + \varepsilon_t V_t \mathbf{u}$. Premultiplying each side by
 133 $V_t^{-1}(\mathbf{I} - \mathbf{u}\mathbf{v})$ and rearranging, we obtain

$$\mathbf{x}_{t+1} \approx \lambda^{-1} \mathbf{K}\mathbf{x}_t + \lambda^{-1}(\mathbf{I} - \mathbf{u}\mathbf{v})\varepsilon_t \mathbf{u}. \quad (\text{B.3})$$

134 In the dynamics of \mathbf{n}_t , the noise term was given by $\varepsilon_t \mathbf{u}V_t$. Here, the noise vector
 135 $\varepsilon_t \mathbf{u}$ is filtered by the matrix $(\mathbf{I} - \mathbf{u}\mathbf{v})$, which removes fluctuations due to
 136 deviations from the stable structure. In Eq. B.3 the projection matrix \mathbf{K}
 137 can also be replaced by the filtered matrix $(\mathbf{I} - \mathbf{u}\mathbf{v})\mathbf{K}$ (Engen et al. 2007). Using

138 this, we get

$$\begin{aligned}\mathbf{x}_{t+1} &\approx (\lambda^{-1}\mathbf{K} - \mathbf{uv})\mathbf{x}_t + \lambda^{-1}(\mathbf{I} - \mathbf{uv})\varepsilon_t\mathbf{u} \\ &= \mathbf{A}\mathbf{x}_t + \mathbf{B}_t.\end{aligned}\tag{B.4}$$

139 By iterating this function we have that

$$\mathbf{x}_t = \mathbf{A}^t\mathbf{x}_0 + \mathbf{B}_{t-1} + \mathbf{A}\mathbf{B}_{t-2} + \mathbf{A}^2\mathbf{B}_{t-3} + \dots$$

The matrix \mathbf{A}^t tends to zero as t increases, with a rate given by λ_2/λ , where λ_2 is the subdominant eigenvalue of \mathbf{K} (Engen et al. 2007). Thus, the term $\mathbf{A}^t\mathbf{x}_0$ tends to zero and the process \mathbf{x}_t approaches a stationary distribution and can be written as

$$\mathbf{x}_t \approx \sum_{i=1}^t \mathbf{A}^{i-1}\mathbf{B}_{t-i},\tag{B.5}$$

144 The expectation of \mathbf{x}_t is $E[\mathbf{x}_t] = \mathbf{0}$ because $E[\varepsilon_t] = \mathbf{0}$. The autocovariance
145 function of X_t can therefore be found from sums of elements in the matrix

146 $E[\mathbf{x}_t \mathbf{x}'_{t+h}]$, where $'$ denotes matrix transposition. This matrix is given by (Engen
 147 et al. 2007)

$$E[\mathbf{x}_t \mathbf{x}'_{t+i}] = \lambda^{-2} \sum_{i=1}^t \mathbf{G}_i \mathbf{S} \mathbf{G}'_{i+h}, \quad (\text{B.6})$$

148 where $\mathbf{G}_t = \mathbf{A}^{t-1}(\mathbf{I} - \mathbf{u}\mathbf{v})$ and $S = E[\varepsilon \mathbf{u} \mathbf{u}' \varepsilon']$. The elements of S are given by
 149 $S_{ij} = \sum_{l=1}^k u_l^2 \text{Cov}(M_{il}, M_{jl})$. Thus, if we know the variance and covariance of
 150 elements in \mathbf{M}_t , together with λ , \mathbf{u} , and \mathbf{v} (obtained from \mathbf{K}), we can find the 151
 autocovariance function of X_t by calculating the matrix in Eq. B.6 and summing its
 elements.

153 The autocorrelation function $\rho_i = \text{Corr}(\Delta \ln N_t, \Delta \ln N_{t+i})$ will tend to zero
 154 as the time lag i increases, but its shape will depend on the type of heterogeneity
 155 and life history of the population. We now show how ρ_i depends on the processes
 156 of $\ln N_t$ and X_t . First, consider the covariance $\text{Cov}(\Delta \ln N_t, \Delta \ln N_{t+i})$. Using
 157 $\Delta \ln N_t = \Delta \ln V_t + \Delta X_t$ (recalling that $\ln N_t \approx \ln V_t + X_t$), we have

$$\begin{aligned}
\text{Cov}(\Delta \ln N_t, \Delta \ln N_{t+i}) &= \text{Cov}(\Delta \ln V_t + \Delta X_t, \Delta \ln V_{t+i} + \Delta X_{t+i}) \\
&= \text{Cov}(\Delta \ln V_t, \Delta \ln V_{t+i}) + \text{Cov}(\Delta \ln V_t, \Delta X_{t+i}) \quad (\text{B.7}) \\
&\quad + \text{Cov}(\Delta X_t, \Delta \ln V_{t+i}) + \text{Cov}(\Delta X_t, \Delta X_{t+i}).
\end{aligned}$$

158 Of the last four covariance terms, the first is zero because the process $\ln V_t$ is
159 approximately white noise with no autocorrelation (Engen et al. 2007). The
160 second covariance is non-zero because ΔX_{t+i} and $\Delta \ln V_t$ depend on the same
161 noise matrix ε_t . The third covariance is zero because we assumed that the noise
162 matrices ε_t are independent, and the two terms do not depend on the same noise
163 matrices ($\Delta \ln V_{t+i}$ depends on ε_{t+i} , whereas ΔX_t depends on $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{t-1}$).
164 The fourth covariance is found by using Eq. B.6, as described above, while the
165 second covariance term of Eq. B.7 is given by

$$\begin{aligned}
\text{Cov}(\Delta \ln V_t, \Delta X_{t+i}) &= \text{Cov}(\lambda^{-1} \mathbf{v} \varepsilon_t \mathbf{u}, X_{t+i+1} - X_{t+i}) \\
&= \text{Cov}(\lambda^{-1} \mathbf{v} \varepsilon_t \mathbf{u}, X_{t+i+1}) - \text{Cov}(\lambda^{-1} \mathbf{v} \varepsilon_t \mathbf{u}, X_{t+i}).
\end{aligned}$$

166 The only elements of X_{t+i+1} or X_{t+i} that enter this covariance are those 167
depending on ε_t . Using Eq. B.5, we can write

168 $\mathbf{x}_{t+i+1} = \mathbf{A}^i \mathbf{B}_t + \mathbf{B}_{t+i} + \mathbf{A} \mathbf{B}_{t+i-1} + \dots$, where only the term $\mathbf{A}^i \mathbf{B}_t$ contains ε_t .

169 Thus,

$$\begin{aligned} \text{Cov}(\lambda^{-1} \mathbf{v} \varepsilon_t \mathbf{u}, X_{t+i+1}) &= \text{Cov}(\lambda^{-1} \mathbf{v} \varepsilon_t \mathbf{u}, \mathbf{e} \mathbf{A}^i \mathbf{B}_t) \\ &= \text{Cov}(\lambda^{-1} \mathbf{v} \varepsilon_t \mathbf{u}, \mathbf{e}[(\lambda^{-1} \mathbf{K})^i - \mathbf{u} \mathbf{v}] \varepsilon_t \mathbf{u}). \end{aligned} \quad (\text{B.8})$$

170 Using this, we find

$$\begin{aligned} \text{Cov}(\Delta \ln V_t, \Delta X_{t+i}) &= \text{Cov}(\lambda^{-1} \mathbf{v} \varepsilon_t \mathbf{u}, \mathbf{e}[(\lambda^{-1} \mathbf{K})^i - \mathbf{u} \mathbf{v}] \varepsilon_t \mathbf{u}) - \text{Cov}(\lambda^{-1} \mathbf{v} \varepsilon_t \mathbf{u}, \mathbf{e}[(\lambda^{-1} \mathbf{K})^{i-1} - \mathbf{u} \mathbf{v}] \varepsilon_t \mathbf{u}) \\ &= \lambda^{-1} [\text{Cov}(\mathbf{v} \varepsilon_t \mathbf{u}, \mathbf{e}(\lambda^{-1} \mathbf{K})^i \varepsilon_t \mathbf{u}) - \text{Cov}(\mathbf{v} \varepsilon_t \mathbf{u}, \mathbf{e}(\lambda^{-1} \mathbf{K})^{i-1} \varepsilon_t \mathbf{u})] \end{aligned} \quad (\text{B.9})$$

171 Together with Eq. B.6, this equation defines the autocovariance function

172 of $\Delta \ln N_t$ from the underlying vital rates, provided that all relevant covariances 173

of \mathbf{M}_t are known.

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